

EINSTEIN–WEYL GEOMETRY, DISPERSIONLESS HIROTA EQUATION AND VERONESE WEBS

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ABSTRACT. We exploit the correspondence between the three-dimensional Lorentzian Einstein–Weyl geometries of the hyper–CR type, and the Veronese webs to show that the former structures are locally given in terms of solutions to the dispersionless Hirota equation. We also demonstrate how to construct hyper–CR Einstein–Weyl structures by Kodaira deformations of the flat twistor space $T\mathbb{CP}^1$, and how show to recover the pencil of Poisson structures in five dimensions illustrating the method by an example of the Veronese web on the Heisenberg group.

1. INTRODUCTION

The notion of three-dimensional Veronese webs appearing in the study of finite dimensional bi-Hamiltonian system is based on existence of one-parameter families of foliations of an open set of \mathbb{R}^3 by surfaces [15, 31, 27]. The same structure underlies the Einstein–Weyl geometry in 2+1 dimensions, where the surfaces are required to be totally geodesics with respect to some torsion-free connection compatible with a conformal structure [6, 16]. The connection between the Veronese webs and Einstein–Weyl geometry has not so far been made, and one purpose of this note is to show that the three-dimensional Veronese webs correspond to a subclass of Einstein–Weyl structures which arise as symmetry reductions of (para) hyper-Hermitian structures in signature $(2, 2)$. This class of Einstein–Weyl structures is called hyper–CR as it admits a hyperboloid of (para) CR structures [11]. In the next Section we shall make use of this correspondence to show that all hyper–CR Einstein–Weyl structures locally arise from solutions of the dispersionless Hirota equation (Theorem 2.1). In Section 3 we shall elucidate the procedure of recovering a hyper–CR Einstein–Weyl structure (and thus also a Veronese web) from the corresponding twistor space. In Section 4 (Theorem 4.1) we give an explicit construction of the bi-Hamiltonian structure in five dimensions from the hyper–CR Einstein–Weyl structures. In this construction the twistor function corresponds to the Casimir of the Poisson pencil. Finally we use the example of a Veronese web on the three-dimensional Heisenberg group to illustrate the procedure of recovering a solution of the dispersionless Hirota equation from a given three-parameter family of twistor curves.

2. EINSTEIN–WEYL STRUCTURES FROM DISPERSIONLESS HIROTA EQUATION

Consider the dispersionless integrable equation of Hirota type [31, 5, 22] (see also [8] for a discussion of its discrete version)

$$(b - a) w_x w_{yz} + a w_y w_{zx} - b w_z w_{xy} = 0, \quad (1)$$

where $w = w(x, y, z)$ is the unknown function on an open set $B \subset \mathbb{R}^3$ and a, b are non-zero constants such that $a \neq b$. This equation arises as the Frobenius integrability condition

$[L_0, L_1] = 0$ for the dispersionless Lax pair of vector fields

$$L_0 = \partial_z - \frac{w_z}{w_x} \partial_x + \lambda a \partial_z, \quad L_1 = \partial_y - \frac{w_y}{w_x} \partial_x + \lambda b \partial_y. \quad (2)$$

This Lax pair is linear in the spectral parameter λ and does not contain derivatives with respect to this parameter. It therefore fits into the formalism of [11], and one expects equation (1) to give rise to an Einstein–Weyl structure on B of hyper-CR type. To construct this structure we could find a linear combination of L_0, L_1 of the form $V_1 - \lambda V_2, V_2 - \lambda V_3$, where $V_i, i = 1, 2, 3$ are vector fields on B , then read off the contravariant conformal structure $V_2 \odot V_2 - V_1 \odot V_3$ and try to solve a system of differential equations for the Einstein–Weyl one-form. We shall instead use another procedure which will give the one-form directly (yet another, more straightforward method applicable to a broad class of dispersionless integrable PDEs has been proposed recently [14]). Extend (2) to a Lax pair of vector fields on a four-manifold $M = B \times \mathbb{R}$, where τ is the coordinate on \mathbb{R}

$$L_0' = L_0 + \partial_\tau, \quad L_1' = L_1.$$

This Lax pair is of the form $L_0' = W_1 - \lambda W_2, L_1' = W_3 - \lambda W_4$, where W_1, W_2, W_3, W_4 are linearly independent vector fields on M . Therefore it defines an anti-self-dual hyper-hermitian conformal structure of neutral signature [10, 12, 7] given in its contravariant form by¹ $W_1 \odot W_4 - W_2 \odot W_3$. Let g be the corresponding covariant form. This hyper-hermitian structure admits a non-null Killing vector $K = \partial/\partial\tau$, and thus the Jones–Tod procedure [17] gives rise to an Einstein–Weyl structure on the space of orbits B of the group action generated by K in M . This is in general given by

$$h = |K|^{-2}g - |K|^{-4}\mathbf{K} \otimes \mathbf{K}, \quad \omega = 2|K|^{-2} *_g (\mathbf{K} \wedge d\mathbf{K}), \quad (3)$$

where $\mathbf{K} = g(K, \cdot)$, $|K|^2 = g(K, K)$, and $*$ _g is the Hodge operator of g . This defines a Weyl structure consisting of a conformal structure $[h] = \{ch, c : B \rightarrow \mathbb{R}^+\}$, and a torsion-free connection D which is compatible with $[h]$ in the sense that

$$Dh = \omega \otimes h.$$

This compatibility condition is invariant under the transformation $h \rightarrow \phi^2 h$, $\omega \rightarrow \omega + 2d(\ln \phi)$, where ϕ is a non-zero function on B .

Applying the Jones–Tod procedure (3) to the Lax pair (2), rescaling the resulting three-metric by $\phi^2 = w_z/(w_x w_y)$, and using (1) to simplify the resulting one-form yields the Weyl structure

$$\begin{aligned} h &= \frac{w_x}{w_y w_z} dx^2 + \frac{a^2 w_y}{(a-b)^2 w_x w_z} dy^2 + \frac{b^2 w_z}{(a-b)^2 w_x w_y} dz^2 \\ &\quad + 2 \frac{a}{(a-b) w_z} dx dy - 2 \frac{b}{(a-b) w_y} dx dz + 2 \frac{ab}{(a-b)^2 w_x} dy dz, \\ \omega &= -\frac{w_{xx}}{w_x} dx - \frac{w_{yy}}{w_y} dy - \frac{w_{zz}}{w_z} dz. \end{aligned} \quad (4)$$

¹The vector fields L_0', L_1' span an \mathbb{RP}^1 worth of surfaces - called α surfaces - through each point of M . The resulting conformal structure $[g]$ is uniquely defined by demanding that these surfaces are totally isotropic and self-dual. The Frobenius integrability condition $[L_0', L_1'] = 0$ modulo L_0' and L_1' is equivalent to vanishing of the self-dual part of the Weyl tensor of $[g]$. In general the distribution L_0', L_1' is defined on the \mathbb{RP}^1 bundle of real projective spinors over M , and the vector fields L_0', L_1' contain derivatives w.r.t the parameter λ . If no such derivatives are present, then $[g]$ contains a metric which is hyper-Hermitian [10, 12], i. e. there exist three real anti-commuting endomorphisms $I, S, T : TM \rightarrow TM$ such that

$$S^2 = T^2 = IST = -I^2 = \mathbf{1},$$

and any $g \in [g]$ is hermitian w.r.t the hyperboloid $a^2 - b^2 - c^2 = 1$ of complex structures $\mathbf{J} = aI + bS + cT$.

A Weyl structure is said to be Einstein–Weyl if the symmetrised Ricci tensor of D is proportional to some metric $h \in [h]$. This conformally invariant condition can be formulated directly as a set of non-linear PDEs on the pair (h, ω) :

$$R_{ij} + \frac{1}{2}\nabla_{(i}\omega_{j)} + \frac{1}{4}\omega_i\omega_j - \frac{1}{3}\left(R + \frac{1}{2}\nabla^k\omega_k + \frac{1}{4}\omega^k\omega_k\right)h_{ij} = 0,$$

where ∇ , R_{ij} , and R are respectively the Levi–Civita connection, the Ricci tensor and the Ricci scalar of h . A direct computation shows that the Weyl structure (4) is Einstein–Weyl iff the dispersionless Hirota equation (1) holds. We must have $w_x w_y w_z \neq 0$ for the conformal structure to be non-degenerate. For example $w = x + y + z$ gives a flat Einstein–Weyl structure, with $\omega = 0$ and h the Minkowski metric on $B = \mathbb{R}^{2,1}$.

According to Gelfand and Zakharevich [15], a Veronese web on a three-manifold B is a one-parameter family of foliations of B by surfaces, such that the normal vector fields to these surfaces through any $p \in B$ form a Veronese curve in $\mathbb{P}(T_p^*B)$. The Veronese web underlying the PDE (1) corresponds to a dual Veronese curve in $\mathbb{P}(T_p B)$

$$\lambda \longrightarrow V(\lambda) = V_1 - 2\lambda V_2 + \lambda^2 V_3,$$

where $\lambda \in \mathbb{RP}^1$, and the vector fields V_i are given by

$$V_1 = ab(aw_y\partial_z - bw_z\partial_y), \quad V_2 = ab(w_y\partial_z - w_z\partial_y), \quad V_3 = bw_y\partial_z - aw_z\partial_y + (a-b)\frac{w_y w_z}{w_x}\partial_x.$$

One of the seminal results of [31] is that all Veronese webs in three dimensions are locally of this form, and thus arise from solutions to equation (1).

We observe that $h(V(\lambda), V(\lambda)) = 0$ for all λ , where h is given by (4), so $V(\lambda)$ determines a pencil of null vectors. The vector fields $V_1 - \lambda V_2$ and $V_2 - \lambda V_3$ (or equivalently the vector fields L_0 and L_1 given by (2)) form an orthogonal complement of $V(\lambda)$. For each $\lambda \in \mathbb{RP}^1$ they span a null surface in B which is totally geodesic with respect to D . The existence of such totally geodesic surfaces for a triple $(B, [h], D)$ is equivalent to the Einstein–Weyl condition [6]. Thus Veronese webs in three dimensions form a subset of Einstein–Weyl structures. The necessary and sufficient condition for an Einstein–Weyl structure to correspond to a Veronese web is that its Lax pair does not contain derivatives w.r.t. λ . We have therefore established

Theorem 2.1. *There is a one to one correspondence between three-dimensional Veronese webs and Lorentzian Einstein–Weyl structure of hyper-CR type. All hyper-CR Einstein–Weyl structures are locally of the form (4), where the function $w = w(x, y, z)$ satisfies equation (1).*

This Theorem, together with results of [19] implies that there is a one to one correspondence between hyper-CR Einstein–Weyl structures and third order ODEs such that certain two fibre preserving contact invariants vanish.

Twistor theory. In the real analytic category one can make use of a complexified setting, where w is a holomorphic function on a complex domain $B_{\mathbb{C}} \subset \mathbb{C}^3$, and the Einstein–Weyl structure is also holomorphic. The twistor space of the PDE (1) is a two-dimensional complex manifold \mathcal{Z} whose points correspond to totally geodesic surfaces in $B_{\mathbb{C}}$. This gives rise to a double fibration picture

$$B_{\mathbb{C}} \xleftarrow{\nu} B_{\mathbb{C}} \times \mathbb{CP}^1 \xrightarrow{\mu} \mathcal{Z}, \tag{5}$$

where \mathcal{Z} arises as a factor space of $B_{\mathbb{C}} \times \mathbb{CP}^1$ by the rank-two distribution $\{L_0, L_1\}$ spanned by (2). The points in $B_{\mathbb{C}}$ correspond to rational curves, called twistor curves, in \mathcal{Z} with self-intersection two, i.e. the normal bundle is $N(l_p) = \mathcal{O}(2)$, where $l_p = \mu \circ \nu^{-1}(p)$, for $p \in B_{\mathbb{C}}$.

The special case of the hyper-CR Einstein-Weyl structures is characterised by the existence of a holomorphic fibration of \mathcal{Z} over a complex projective line [11]. The existence of this fibration is a consequence of the absence of vertical $\partial/\partial\lambda$ terms in the Lax pair (2). The real manifold B (and thus the real solutions to (1)) corresponds to twistor curves invariant under an anti-holomorphic involution $\tau : \mathcal{Z} \rightarrow \mathcal{Z}$ which fixes an equator of each twistor curve.

Hierarchies. Equation (1) can be embedded into an infinite hierarchy of integrable PDEs, which arises from the distribution

$$L_i = \partial_i - \frac{\partial_i w}{\partial_x w} \partial_x + \lambda a_i \partial_i, \quad i = 0, 1, \dots$$

and is given by $[L_i, L_j] = 0$, or

$$(a_i - a_j) \partial_x w \partial_i \partial_j w + a_j \partial_i w \partial_j \partial_x w - a_i \partial_j w \partial_i \partial_x w = 0, \quad (\text{no summation}). \quad (6)$$

Here $\partial_i = \partial/\partial x^i$ and a_i are distinct constants. The function $w = w(x, x_0, x_1, \dots)$ depends on an infinite number of independent variables, but the Cauchy data for the overdetermined system (6) only depends on functions of two-variables which is the same functional degrees of freedom as in (1).

3. HYPER-CR EQUATION AND KODAIRA DEFORMATION THEORY

In [11] it was shown that all Lorentzian hyper-CR Einstein-Weyl spaces on an open set $B \subset \mathbb{R}^3$ are locally of the form

$$h = (dY + H_X dT)^2 - 4(dX - H_Y dT)dT, \quad \omega = H_{XX} dY + (H_X H_{XX} + 2H_{XY})dT, \quad (7)$$

where $H = H(X, Y, T)$ satisfies the hyper-CR equation [9, 28, 13, 26, 21, 4]

$$H_{XT} - H_{YY} + H_Y H_{XX} - H_X H_{XY} = 0. \quad (8)$$

Combining this result with Theorem 2.1 suggests that PDEs (1) and (8) are equivalent, as there exists a local diffeomorphism of B mapping (4) to (7). It can be shown that there is no point equivalence between these equations [5], but this does not rule out the existence of a Backlund type transformation between them which involves the second derivatives of the dependent variables (a transformation of this type connects the first and the second Plebanski heavenly equations [12]) or a Legendre transformation of the type recently analysed by Bogdanov [2]. A procedure recovering a solution of the Hirota equation (1) from the hyper-CR equation (8) will be illustrated (at the end of Section 4) by the example of the Veronese web on the Heisenberg group.

In this section we shall simplify the procedure of recovering the solution to (8) and the corresponding Einstein-Weyl structure from the twistor data [11]. Equation (8) is equivalent to $[L_0, L_1] = 0$, where the Lax pair is²

$$L_0 = \partial_Y - \lambda(\partial_T + H_Y \partial_X), \quad L_1 = \partial_X - \lambda(\partial_Y + H_X \partial_X). \quad (9)$$

The ring of twistor functions spanning the kernel of this Lax pair is (ψ, λ) , where

$$\psi = T + \lambda Y + \lambda^2 X + \lambda^3 H + \lambda^4 \psi_4 + \dots = \sum_{i=0}^{\infty} \lambda^i \psi_i, \quad (10)$$

and the functions ψ_i depend on (X, Y, T) but not on λ . The recursion relations connecting ψ_{i+1} to ψ_i arise from equating coefficients of λ in $L_0 \psi = L_1 \psi = 0$ to zero. The consistency conditions for these relations imply that any of the functions ψ_i satisfies

$$(\partial_X \partial_T - \partial_Y^2 + H_Y \partial_X^2 - H_X \partial_X \partial_Y) \psi_i = 0.$$

²This differs from the Lax pair of [11] by replacing λ by $1/\lambda$.

Note that this is different than a linearisation of (8).

In the real analytic setup, where the twistor methods can be applied, the conformal structure $[h]$ arises from defining a null vector at $p \in B_{\mathbb{C}}$ to be a section of a normal bundle $N(l_p)$ which vanishes at one point on the twistor line $l_p \cong \mathbb{CP}^1$ in the twistor space \mathcal{Z} to second order. To define the connection D , define a direction at $p \in B_{\mathbb{C}}$ to be the one-dimensional space of sections on $N(l_p)$ which vanish at two points. The one-dimensional family of curves vanishing at these two points corresponds to a geodesics in $B_{\mathbb{C}}$. In the limiting case when these two points coincide this geodesic is null w.r.t $[h]$ in agreement with the definition of the Weyl structure [16]. The procedure of recovering both D and $[h]$ from a family of curves in \mathcal{Z} is explicit, but rather involved [23]. This can be vastly simplified by using the local form (7): given a three-parameter family of sections of $\mathcal{Z} \rightarrow \mathbb{CP}^1$

$$\lambda \longrightarrow (\lambda, \psi(\lambda))$$

choose a point on the base of the fibration $\mathcal{Z} \rightarrow \mathbb{CP}^1$, say $\lambda = 0$, and parametrise this family by coordinates (X, Y, T) where

$$T = \psi|_{\lambda=0}, \quad Y = \frac{\partial \psi}{\partial \lambda}|_{\lambda=0}, \quad X = \frac{1}{2} \frac{\partial^2 \psi}{\partial \lambda^2}|_{\lambda=0}. \quad (11)$$

Expanding the twistor function ψ in a Laurent series in λ gives (10), and we can read off $H = H(X, Y, T)$ from the coefficient of λ^3 . The Einstein–Weyl structure is now given by (7), and we did not have to follow the procedure of [23].

For example, expanding the Nil twistor function [11] (which depends on a constant parameter ϵ , and in the limit $\epsilon \rightarrow 0$ reduces to a section $\lambda \rightarrow T + \lambda Y + \lambda^2 X$ of the undeformed twistor space $T\mathbb{CP}^1$)

$$\psi = T + \lambda Y - \frac{\lambda}{\epsilon} \ln(1 - \lambda \epsilon X) \quad (12)$$

in a power series in λ and comparing with (10) gives $H = \epsilon X^2/2$. Using (7) we find the corresponding one-parameter family of deformations of the flat Einstein–Weyl structure is given by Lorentzian Nil geometry

$$h = (dY + \epsilon X dT)^2 - 4dX dT, \quad \omega = \epsilon(dY + \epsilon X dT). \quad (13)$$

We shall come back to this example at the end of Section 4.

Deformation theory. Let $[\psi : \pi_0 : \pi_1]$ be homogeneous holomorphic coordinates on the twistor space $\mathcal{Z} = T\mathbb{CP}^1$ corresponding to the flat Einstein–Weyl structure with $H = 0$. Here $[\pi_0 : \pi_1]$ are homogeneous coordinates on the base \mathbb{CP}^1 and $\lambda = \pi_0/\pi_1$. Cover \mathcal{Z} by two open sets \mathcal{U} and $\tilde{\mathcal{U}}$ with $\pi_1 \neq 0$ on \mathcal{U} and $\pi_0 \neq 0$ on $\tilde{\mathcal{U}}$. Following the steps of Penrose’s non-linear graviton construction [29] we construct twistor spaces corresponding to non-trivial Einstein–Weyl spaces by deforming the patching relation between \mathcal{U} and $\tilde{\mathcal{U}}$. The Kodaira theorem [18] guarantees that the deformations preserve the three-parameter family of curves and the deformed twistor space still gives rise to a three-dimensional manifold $(B_{\mathbb{C}}, [h], D)$ with Einstein–Weyl structure.

The hyper-CR deformations preserve the fibration of \mathcal{Z} over \mathbb{CP}^1 . Thus we deform the patching relations such that π_0/π_1 is preserved on the overlap. The infinitesimal deformations of this kind are generated by elements of $H^1(\mathcal{Z}, \Theta)$, where Θ denotes a sheaf of germs of holomorphic vector fields. Let a vector field

$$\mathcal{Y} = f \frac{\partial}{\partial \psi} + g \left(\pi_0 \frac{\partial}{\partial \pi_0} + \pi_1 \frac{\partial}{\partial \pi_1} \right)$$

defined on an overlap of \mathcal{U} and $\tilde{\mathcal{U}}$ define a class in $H^1(\mathcal{Z}, \Theta)$. Here the functions f and g are holomorphic and homogeneous of degree two and zero respectively on the overlap. The finite deformation $\tilde{\psi} = \tilde{\psi}(\psi, \lambda, \epsilon)$ is given by integrating

$$\frac{d\psi}{d\epsilon} = f, \quad \frac{d\pi_0}{d\epsilon} = g\pi_0, \quad \frac{d\pi_1}{d\epsilon} = g\pi_1.$$

It is possible that the assumption about real-analyticity can be dropped by employing the techniques of [24] or some modification of the dispersionless $\bar{\partial}$ approach [3], but that remains to be seen.

4. BI-HAMILTONIAN STRUCTURE

Recall that a Poisson structure on a manifold U is a bivector $P \in \Lambda^2(TU)$ such that the associated Poisson bracket $\{f, g\} := P \lrcorner (df \wedge dg)$ satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$$

for all smooth functions f, g, h on U . If we regard $U \subset \mathbb{R}^n$ as an open set, and adapt local coordinates such that

$$P = \sum_{\alpha, \beta=1}^n P^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta},$$

then the Poisson bracket takes the form

$$\{f, g\} = \sum_{\alpha, \beta=1}^n P^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial g}{\partial x^\beta}.$$

If n is odd, the Poisson structure necessarily admits at least one Casimir, i.e. a non-constant function C such that $\{C, f\} = 0$ for all functions f on U .

A bi-Hamiltonian structure P^λ on U consists of two Poisson structures P_0 and P_1 such that the pencil of Poisson brackets

$$\{f, g\}_\lambda := \{f, g\}_0 + \lambda \{f, g\}_1$$

satisfies the Jacobi identity for all values of the parameter λ (see e.g. [20, 25, 1]). We shall assume that P^λ has exactly one Casimir function C^λ for each λ . Thus, regarding the $P^\lambda \in \Gamma(\Lambda^2(TU))$ as a map from T^*U to TU , we find that this map has a one-dimensional kernel. Therefore, for each $\lambda \in \mathbb{RP}^1$, the manifold U is foliated by four-dimensional symplectic leaves of P^λ , i. e. there exists a rank four integrable distribution $\mathcal{D}_\lambda \subset TU$ such that $\mathcal{D}_\lambda = \ker(dC^\lambda)$. This distribution is annihilated by a λ -dependent one-form

$$\mathbf{e}(\lambda) := (P^\lambda \wedge P^\lambda) \lrcorner \Omega,$$

where Ω is some fixed volume form on U , and \lrcorner denotes its contraction with a section of $\Lambda^4(TU)$. Therefore $\mathbf{e}(\lambda)$ is quadratic in λ , and we can write

$$\mathbf{e}(\lambda) = e^3 + \lambda e^2 + \lambda^2 e^1 \tag{14}$$

for some one-forms e^i . The Pfaff theorem implies that $dC^\lambda \in \text{span}(\mathbf{e}(\lambda))$.

The intersection of all distributions \mathcal{D}_λ as $\lambda \in \mathbb{RP}^1$ varies is an integrable rank-two distribution which we shall call \mathcal{D} . Thus

$$\mathcal{D} = \ker e^1 \cap \ker e^2 \cap \ker e^3 = \bigcap_{\lambda} \ker(\mathbf{e}(\lambda)) \subset TU$$

which is integrable, as an intersection of tangent bundles to the symplectic leaves of the Casimir C^λ . Therefore $\mathcal{D} \subset \mathcal{D}_\lambda \subset TU$. We shall assume that all these distributions have constant ranks. The two dimensional distribution $\mathcal{D}_\mathcal{Z} := \mathcal{D}_\lambda / \mathcal{D}$ is defined on the three

dimensional quotient $B := U/\mathcal{D}$. This distribution is integrable, and it defines the structure of a Veronese web on B . This web is a projection of symplectic leaves of P^λ to B .

This is essentially the construction of [15]. In [15] an inverse construction of a bi-Hamiltonian structure from a Veronese web is also given. This construction does not appear to be explicit, but it has been shown [30] that a bi-Hamiltonian structure on U can be recovered from a Veronese web on B .

We shall now give a simple algorithm for recovering the bi-Hamiltonian structure from any hyperCR Einstein–Weyl structure which, by Theorem 2.1, is equivalent to a Veronese web. In our procedure the distribution \mathcal{D}_Z is identified with the twistor distribution (2) given by the span of the Lax pair L_0 and L_1 which (in the complexified setting) underlies the double fibration picture (5). The one forms e^i in (14) define the conformal structure

$$h = e^2 \otimes e^2 - 2(e^1 \otimes e^3 + e^3 \otimes e^1)$$

and the Frobenius condition $\mathbf{e} \wedge d\mathbf{e} = 0$ is equivalent to the hyper–CR Einstein–Weyl equations [11].

Before formulating the next Theorem recall that the conformal signature of signature $(2, 1)$ on a three–dimensional manifold B is equivalent to the existence of an isomorphism

$$TB \cong \mathbb{S} \odot \mathbb{S}, \quad (15)$$

where \mathbb{S} is a rank two real symplectic vector bundle over B . In concrete terms (15) identifies vectors with two by two symmetric matrices (or equivalently, with homogeneous quadratic polynomials in two variables). The conformal structure is then defined by declaring a vector to be null iff the corresponding matrix has rank one (or equivalently, if the corresponding homogeneous polynomial has a repeated root). The isomorphism (15) is a particular example of a $GL(2, \mathbb{R})$ structure: the group $GL(2, \mathbb{R})$ acts on the fibres of \mathbb{S} , and induces Lorentz rotations and conformal rescalings on vectors in B as $\mathfrak{gl}(2, \mathbb{R}) \cong \mathfrak{so}(2, 1) \oplus \mathbb{R}$.

Let (p_0, p_1) be local coordinates on the fibres of the dual vector bundle \mathbb{S}^* .

Theorem 4.1. *Let $(B, [h], D)$ be a hyper–CR Einstein–Weyl structure and let (L_0, L_1) be the commuting twistor distribution (Lax pair) spanning a pencil of null and totally geodesic surfaces through each point $p \in B$. Then*

$$P^\lambda = L_0 \wedge \frac{\partial}{\partial p_0} + L_1 \wedge \frac{\partial}{\partial p_1}. \quad (16)$$

is a bi–Hamiltonian structure on the five–dimensional manifold \mathbb{S}^ , where \mathbb{S}^* is a real rank two bundle over B such that (15) is the canonical isomorphism given by the conformal structure $[h]$ on B .*

Proof. Consider three linearly independent vector fields (V_1, V_2, V_3) on B such that the conformal structure in the Einstein–Weyl structure is represented by the contravariant form $V_2 \odot V_2 - V_1 \odot V_3$. Let $\zeta_\lambda \subset B$ be a two–dimensional totally geodesic surface through $p \in B$ corresponding to $\lambda \in \mathbb{RP}^1$ and let (p_0, p_1) be coordinates on the fibres of the cotangent bundle $T^*\zeta_\lambda$. The tangent space $T_p\zeta_\lambda$ is spanned by the twistor distribution $L_0 = V_1 - \lambda V_2, L_1 = V_2 - \lambda V_3$ (any twistor distribution underlying a hyper–CR Einstein–Weyl space can be put in this form, by taking a linear combination of the spanning vector with λ –independent coefficients) and so the canonical Poisson structure on $T^*\zeta_\lambda$ is

$$L_0 \wedge \frac{\partial}{\partial p_0} + L_1 \wedge \frac{\partial}{\partial p_1}.$$

We now extend this to a bi–Hamiltonian structure on $B \times \mathbb{R}^2$. To do it we need to canonically identify the tangent spaces to different null totally geodesic surfaces in B and show that the union of cotangent bundles $\bigcup_{\lambda \in \mathbb{RP}^1} T^*\zeta_\lambda$ is isomorphic to \mathbb{S}^* . To establish the latter,

observe that under the isomorphism (15) vectors tangent to ζ_λ correspond to quadratic polynomials with one of the roots equal to λ : represent the basis (V_1, V_2, V_3) of $T_p B$ as a symmetric matrix of vector fields V_{AB} with $A, B = 0, 1$ and $V_{00} = V_1, V_{01} = V_{10} = V_2, V_{11} = V_3$, and the twistor distribution is $L_A = \pi^B V_{AB}$ where $\pi^B = [\pi^0, \pi^1]$ are homogeneous coordinates on \mathbb{RP}^1 such that $\pi^1/\pi^0 = -\lambda$. Thus any vector tangent to ζ_λ is $W = \mu^A L_A$ for some $\mu^A = [\mu^0, \mu^1]$. The two by two matrices corresponding to such vectors are $W = \mu \odot \pi$, with π^A fixed and μ^A varying. Equivalently vectors tangent to ζ_λ correspond to homogeneous polynomials with common root $[\pi^0, \pi^1]$. Dividing out by the factor containing this root yields linear homogeneous polynomials, i.e. an isomorphism $T\zeta_\lambda \cong \mathbb{S}|_{\zeta_\lambda}$. To obtain the canonical identification of all tangent spaces $T_p \zeta_\lambda$ as λ varies we simply evaluate the twistor distribution at different values of λ : Let ζ_1 and ζ_2 be totally geodesic null surfaces corresponding to λ_1 and λ_2 respectively. The isomorphism $\phi_{12} : T\zeta_1 \rightarrow T\zeta_2$ is defined on the basis vectors by

$$\phi_{12}(L_0|_{\lambda_1}) = L_0|_{\lambda_2}, \quad \phi_{12}(L_1|_{\lambda_1}) = L_1|_{\lambda_2}.$$

Therefore the fibre coordinates (p_0, p_1) can be used unambiguously for all λ and the Poisson structure on $T^*\zeta_\lambda$ extends to a Poisson pencil³ (16) on $\bigcup_{\lambda \in \mathbb{RP}^1} \mathbb{S}^*|_{\zeta_\lambda} = \mathbb{S}^*$.

For any Poisson structure given by a bivector P the Jacobi identity reduces to

$$\sum_{\delta=1}^n P^{\delta\gamma} \frac{\partial P^{\alpha\beta}}{\partial x^\delta} + P^{\delta\beta} \frac{\partial P^{\gamma\alpha}}{\partial x^\delta} + P^{\delta\alpha} \frac{\partial P^{\beta\gamma}}{\partial x^\delta} = 0 \quad \text{for all } \alpha, \beta, \gamma = 1, \dots, n.$$

This set of conditions holds, with $n = 5$, for $P^\lambda = P = P_0 - \lambda P_1$ regardless of a particular form of the Lax pair (L_0, L_1) , as long as $[L_0, L_1] = 0$ because in this case the Frobenius theorem can be used to put the Poisson bi-vector in the canonical form.

If the vector fields of the Lax pair are linear in the parameter λ - which is the case for Lax pairs underlying hyperCR Einstein–Weyl structures - then the bivector (16) defines a pencil of compatible Poisson structures. □

In the construction of the Poisson pencil (16) the Casimir C^λ is a twistor function which we called ψ in formula (10), and

$$\mathcal{D}_Z = \text{span}(L_0, L_1), \quad \mathcal{D} = \text{span}(\partial/\partial p_0, \partial/\partial p_1).$$

Using the explicit form of the twistor distribution (2) yields the following

Corollary 4.2. *Let $w = w(x, y, z)$ be a function on B with continuous second derivatives. The Poisson brackets*

$$P_0 = \begin{pmatrix} 0 & 0 & 0 & -w_z/w_x & -w_y/w_x \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ w_z/w_x & 0 & -1 & 0 & 0 \\ w_y/w_x & -1 & 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & -a & 0 & 0 \\ 0 & -b & 0 & 0 & 0 \end{pmatrix}.$$

define a compatible Poisson pencil if and only if the function $w = w(x, y, z)$ satisfies the dispersionless Hirota equation (1).

³This agrees with the Gelfand–Zakharevich construction [15], where the existence of the isomorphism $T^*B = \mathbb{S}^* \odot \mathbb{S}^*$ is postulated. The authors consider

$$T^*B \otimes \mathbb{S}^* = (\mathbb{S}^* \odot \mathbb{S}^* \odot \mathbb{S}^*) \oplus (\mathbb{S}^* \otimes \Lambda^2(\mathbb{S}^*)).$$

The 5-manifold supporting the Poisson pencil arises as the second factor in this decomposition, i.e. the total space of $\mathbb{S}^* \rightarrow B$ as the line bundle $\Lambda^2(\mathbb{S}^*)$ is trivialised.

To obtain a bi-Hamiltonian system, the Hamiltonian function needs to be selected as one of the coefficients in the expansion of the Casimir C^λ in the powers of λ . It follows from the general theory that all coefficients of this expansion are in involution with respect to both Poisson structures P_0 and P_1 . Thus the coordinate functions on B are first integrals of the Hamiltonian flow, and the integral curves of this flow are vertical to B . They are of the form $p_k(t) = \omega_k(x, y, z)t + \rho_k(x, y, z)$, where $k = 0, 1$, and ω_k, ρ_k are known functions on B whose form depends on the choice of the Hamiltonian. This simple form of solution curves reflects the fact that the Gelfand–Zakharevich construction of Veronese webs puts the system in the action–angle coordinates.

Example. The Heisenberg group. We have already shown that the Heisenberg group carries an Einstein–Weyl structure of Lorentzian signature given by (13). The twistor distribution (9) defining the Veronese web is

$$L_0 = \partial_Y - \lambda \partial_T, \quad L_1 = \partial_X - \lambda(\partial_Y + \epsilon X \partial_X). \quad (17)$$

The metric h defining the conformal structure is left-invariant with the Killing vectors generated by the right invariant vector fields

$$R_Y = \partial_Y, \quad R_T = \partial_T, \quad R_X = \partial_X - \epsilon T \partial_Y$$

forming the Heisenberg Lie algebra

$$[R_X, R_T] = \epsilon R_Y, \quad [R_X, R_Y] = 0, \quad [R_T, R_Y] = 0.$$

The Einstein–Weyl structure is only invariant under the abelian subalgebra spanned by R_Y and R_T as the Lie derivative of the twistor distribution along R_X is non-zero. The Poisson pencil is also not invariant under the Heisenberg action, which is best seen by writing it in terms of the left-invariant vector fields (which commute with the right-invariant vector fields) as

$$P^\lambda = (L_Y \wedge \partial_{p_0} + L_X \wedge \partial_{p_1}) - \lambda((L_T + \epsilon X L_Y) \wedge \partial_{p_0} + (L_Y + \epsilon X L_X) \wedge \partial_{p_1})$$

and $\mathcal{L}_{R_X} P^\lambda = \epsilon \lambda P^0 \neq 0$. We note that there is one Poisson bracket in the pencil (corresponding to $\lambda = 0$) which is left-invariant.

We finish the discussion of this example by presenting the solution to the dispersionless Hirota equation (1) corresponding to the Heisenberg Veronese web (17). The procedure we shall use elucidates two different parametrisations of twistor curves leading to equations (8) and (1) respectively. In case of the hyper-CR equation we choose a point, say $\lambda = 0$, on the base of the fibration $\mathcal{Z} \rightarrow \mathbb{CP}^1$ (here we again work in the complexified setting), and parametrise the curve l_p given by $\lambda \rightarrow (\lambda, \psi(\lambda))$ corresponding to $p \in B$ by the position of its intersection with the fibre over $\lambda = 0$ and the first two jet coordinates as in (11). The third jet then defines the dependent variable H in equation (8). This can be seen from formula (12).

In case of the dispersionless Hirota equation we instead choose four distinct points $\lambda_1, \lambda_2, \lambda_3$ and λ_4 on the base. The intersections of the curve with the three fibres over $\lambda_1, \lambda_2, \lambda_3$ give, up to a reparametrisation, the coordinates (x, y, z) . The intersection with the fibre over λ_4 then defines the dependent variable w . We can use the Möbius transformation to map $(\lambda_1, \lambda_2, \lambda_3)$ to $(0, \infty, 1)$. Thus we expect the coordinate T in the hyper-CR equation to map directly to coordinate x in the dispersionless Hirota equation. This is the twistorial interpretation of the original procedure of Zakharevich where the two-dimensional

distribution $\mathcal{D}_{\mathcal{Z}}$ defining the Veronese web is

$$\begin{aligned}\mathcal{D}_{\mathcal{Z}}(\lambda) = & \ker \left((\lambda_1 - \lambda_4)(\lambda - \lambda_2)(\lambda - \lambda_3)w_x dx \right. \\ & + (\lambda_2 - \lambda_4)(\lambda - \lambda_1)(\lambda - \lambda_3)w_y dy \\ & \left. + (\lambda_3 - \lambda_4)(\lambda - \lambda_1)(\lambda - \lambda_2)w_z dz \right).\end{aligned}$$

Comparing this with $\mathcal{D}_{\mathcal{Z}}(\lambda) = \text{span}(L_0, L_1)$ given by the twistor distribution (17) and using the Möbius freedom described above yields $\mathcal{D}_{\mathcal{Z}}(0) = \ker(dx)$, $\mathcal{D}_{\mathcal{Z}}(1) = \ker(dz)$, $\mathcal{D}_{\mathcal{Z}}(\infty) = \ker(dy)$, or equivalently

$$d\psi|_{\lambda=0} \in \text{span}(dx), \quad d\psi|_{\lambda=1} \in \text{span}(dz), \quad d\psi|_{\lambda=\infty} \in \text{span}(dy),$$

where ψ is the twistor function (12). These formulae define the coordinates up to reparametrizations $(x, y, z) \rightarrow (\hat{x}(x), \hat{y}(y), \hat{z}(z))$, and we choose

$$x = T, \quad y = Xe^{-\epsilon Y}, \quad z = (1 - \epsilon X)e^{-\epsilon(T+Y)}.$$

The solution $w(x, y, z)$ of the dispersionless Hirota equation (1) can now be read-off from $\mathcal{D}_{\mathcal{Z}}(\lambda_4) = \ker(dw)$ which yields $w = (1 - \epsilon\lambda_4 X)e^{-\epsilon(\lambda_4^{-1}T+Y)}$, or after transforming to the new variables and reabsorbing some constants by rescaling (x, y, z) ,

$$w = ye^{ax} + ze^{bx},$$

where $\lambda_4 = 1 - b/a$.

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